Jan 26, 2017

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For a mapping $f: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$, there are two concepts of continuity.

Continuity at $x_0 \in X$: $\forall\ V \in \mathcal{T}_Y$ with $f(x_0) \in V$
$\exists\ U \in \mathcal{T}_X$ with $x_0 \in U$, $U \subset f^{-1}(V)$

Continuity (everywhere): $\forall\ V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$.

Example. Dirichlet function $f: \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

* it is discontinuous everywhere, if
$$f: (\mathbb{R}, \mathcal{T}_{std}) \longrightarrow (\mathbb{R}, \mathcal{T}_{std})$$
* it is continuous everywhere if
$$f: (\mathbb{R}, \underbrace{\mathcal{P}(\mathbb{R})}) \longrightarrow (\mathbb{R}, \text{any})$$
so large to contain any $f^{-1}(V)$.

* it is also continuous everywhere if
$$f: (\mathbb{R}, \text{any}) \longrightarrow (\mathbb{R}, \underbrace{\{\phi, \mathbb{R}\}}_{\text{So small}})$$

$f^{-1}(\phi) = \phi$, $f^{-1}(\mathbb{R}) = \mathbb{R}$, must in any topology

Fact. Continuity is not only about f,
it is about the topologies

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Example. $X = \left\{ \begin{matrix} \text{continuous functions} \\ [a,b] \longrightarrow \mathbb{R} \end{matrix} \right\}$

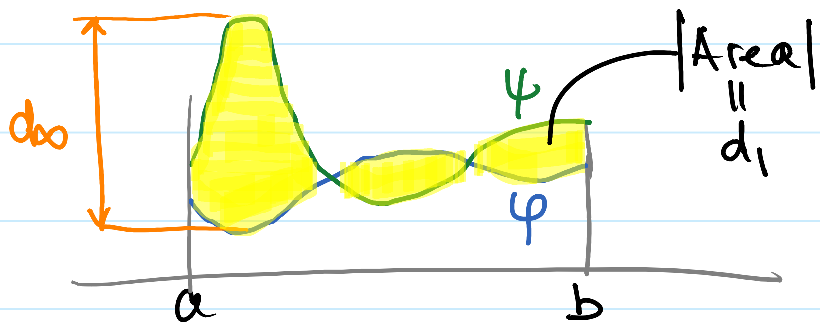
and consider $id : X \longrightarrow X$

① L_1-topology $\mathcal{T}_1$ by the metric

$$d_1(\varphi,\psi) = \int_a^b |\varphi(t) - \psi(t)|\, dt$$

② Uniform Topology $\mathcal{T}_\infty$ by

$$d_\infty(\varphi,\psi) = \sup_{t\in[a,b]} |\varphi(t) - \psi(t)|$$



Fact. $id : (X, \mathcal{T}_\infty) \longrightarrow (X, \mathcal{T}_1)$ is continuous

Idea of proof.

If $d_\infty(\varphi,\psi) < \delta = \frac{\varepsilon}{b-a}$

then $d_1(\varphi,\psi) < \int_a^b \delta\, dt = \varepsilon$

What about

$id : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_\infty)$?

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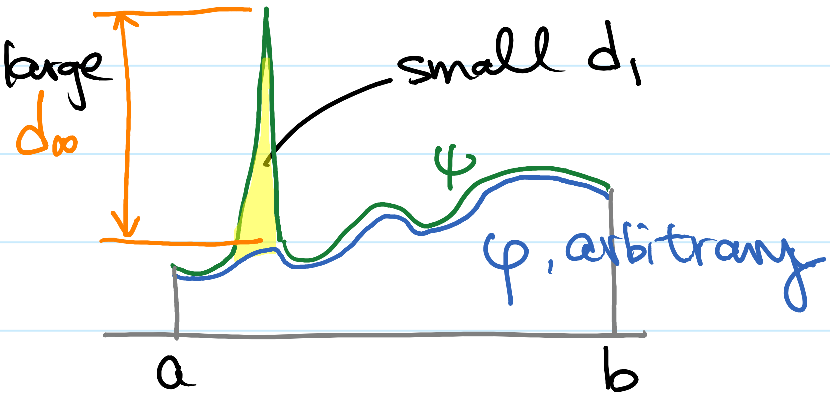
Fact. $\mathrm{id} : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_\infty)$ is not continuous at everywhere (i.e., $\forall \varphi$)

Idea.



Metric argument. For any $\varphi \in X$

$$\exists\ \varepsilon = 1 \quad \forall\ \delta > 0, \quad \text{take } \tfrac{1}{n} < \delta$$

Construct a function $\psi \in X$ which equals φ everywhere except a "triangle" of base $1/2n$ and height 1. Then

$$d_1(\psi, \varphi) = \text{area} = \frac{1}{n} < \delta$$

but

$$d_\infty(\mathrm{id}(\psi), \mathrm{id}(\varphi))$$

$$\|$$

$$\sup_{t \in [a,b]} |\psi(t) - \varphi(t)| > 1 = \varepsilon$$

Exercise. Argument in terms of topology

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Equivalences of Continuity

Given $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ with bases $\mathcal{B}_X, \mathcal{B}_Y$ for $\mathcal{T}_X, \mathcal{T}_Y$ respectively

① f is continuous at x $\forall x \in X$

$\Updownarrow$ known

② $\forall V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$

trivial $\Downarrow$ $\Uparrow$ by taking union

③ $\forall B \in \mathcal{B}_Y$, $f^{-1}(B) \in \mathcal{T}_X$ (not $\mathcal{B}_X$)

$\Downarrow$ See below

④ $\forall E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$

$\Downarrow$ Obvious, use $E = f^{-1}(F)$

⑤ $\forall F \subset Y$, $\overline{f^{-1}(F)} \subset f^{-1}(\overline{F})$

$\Downarrow$ if H is closed, then use $F = H = \overline{H}$

⑥ $\forall$ closed $H \subset Y$, $f^{-1}(H) \subset X$ is closed

Taking complement ② $\Leftrightarrow$ ⑥

② $\Rightarrow$ ④

Let $f(x) \in f(\overline{E}) \overset{?}{\subset} \overline{f(E)}$

for arbitrary $x \in \overline{E}$

and any $V \in \mathcal{T}_Y$ with $f(x) \in V$

Need $V \cap f(E) \neq \emptyset$

i.e., find $e \in E$ $f(e) \in V$

$f^{-1}(V) \in \mathcal{T}_X$ and $x \in \overline{E}$, $\therefore f^{-1}(V) \cap E \neq \emptyset$ gives